Observations on the Maximum Power Transfer Theorem

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***Abstract*—In order tor the Maximum Power Transfer Theorem (MPTT) to be applied in practical systems, two additional constraints or extensions should be invoked.**

1) **The power source must be power limited**

2) **The system must be operating at its maximum power level.**

The first constraint arises from the term "maximum power", which implies that a maximum exists. The second constraint arises when we analyze a system operating at its maximum power level to determine its source impedance. We arrive at these conclusions after discovery two phenomena – the concept of an "apparent source resistance", different from any actual, loss resistance, and the concept of a source resistance which is non-dissipative.

We provide an example of a practical, power limited system, which we analyze at its maximum available power level to determine the "apparent" source impedance. We also introduce the concept of a non-dissipative source resistance and provide the procedure for measuring its value using the well-known principles normally associated with the equivalent circuits of Thévenin or Norton. The results of the analysis will show that the system is matched and yet operates at high efficiency- clearly demonstrating the inadequacy of the traditional interpretation of the MPTT and the Thévenin and Norton equivalent circuits.

I. Introduction

The Maximum Power Transfer Theorem:

> *The maximum power will be absorbed by one network from another joined to it at two terminals, when the impedance of the receiving network is varied, if the impedance looking into the two networks at the junction are conjugates of each other* [1].

For the past 130 years electrical engineers have largely ignored the implications of the MPTT. We have ignored the fact that the MPTT strongly implies that the source of the power being generated and delivered to an electrical load is a power limited source. Furthermore, we have ignored the fact that the MPTT implies that the maximum power transfer is limited to an efficiency of 50% maximum, because the source impedance (losses) are inherently equal to the power absorbed by the load.

Twentieth century wording of the MPTT uses the phrase "conjugates of each other". We interpret that as requiring that the pair of complex impedances be the same except for the sign of the imaginary components. For a DC system, as will be analyzed here, the MPTT only requires a match between the source and load resistances.

In our analysis of the MPTT and its implications we will first define a simple steam turbine prime mover with gearbox and a DC generator with a variable resistive load.

Next, we will create a math model of that system and test it to determine that it correctly models our system.

Finally, we will exercise our math model to analyze both the explicit and implicit constraints of the MPTT, including the concept of a power-limited system and the concept of a non-dissipative resistance.

During the course of these analyses, having discovered the phenomenon of the non-dissipative resistance, we learn how to measure the "apparent" source resistance of the prime-mover and generator combination using both Norton and Thévenin equivalent circuits. We then find that we can demonstrate both an impedance match and efficiency greater than 50%.

In view of the rather startling revelations encountered, we will devote an entire section to the discussion of our results followed by a more sophisticated derivative analysis and, finally, a summary of our results and conclusions – especially in those areas which call for extensions to MPTT.

In order to show that the basic principles are equally applicable to non-electric systems, we define an analogous system using a viscous damper load to replace the DC generator and load. Our results from an analysis of that system show clearly that the same principles apply as were found in the case of the DC generator and load.

II. Objectives

The objectives of this analysis are twofold. The first objective is to demonstrate the apparent source impedance of a system under conditions of a conjugately matched load using a mathematical model and numerical examples. The second objective is to identify and highlight those unique conditions that indicate a conjugate match in such a system.

Inevitably, some readers will ask, why use an electro-mechanical system to explore a theorem known and used primarily in the electrical world? The primary difficulty in the electrical world is the concept of a *power-limited* system. From a pure electrical circuit perspective, power limitations are not an intrinsic part of the model. If you put a 3 ohm resistor across a 12 volt car battery there will be 4 amperes current flow, 48 watts dissipated in the resistor. If the resistor is a 10 watt resistor, it will be destroyed. That is the kind of power-limitation that is common in electrical circuits. In contrast, everyone seems to understand how mechanical systems are limited. If one horse cannot pull the plow, then you need two horses. If a 36 HP Volkswagen cannot climb a steep hill, then perhaps you need a 250 HP Buick for the job. People naturally understand those kinds of power limitations – they do not understand electrical circuits having power limitations, although they do understand that electrical components are burned out through error. For a system to be power-limited is quite different than burning out electrical components or breaking mechanical components. A *power-limited* system is a system with a defined maximum available power output that it is incapable of exceeding.

Considering that most large scale electrical systems draw their energy from prime-movers and that prime-movers are inherently power-limited, it seems reasonable to model a prime-mover/generator combination. Furthermore, it will prove to be a trivial step to later convert that model to an all-mechanical system in order to extend our findings to the non-electrical world.

Another advantage of modeling the system with a turbo-electric system is that we will find that prime-movers have a built in, constant, non-dissipative, source impedance when operating at their maximum available power.

III. System Description

Our hypothetical system consists of three separate elements – a steam turbine prime-mover rated at approximately 75 horsepower (HP), a 10:1 speed reduction gearbox, and a Direct Current electrical generator. All three elements of the system are directly connected via appropriate drive shafts. The steam turbine converts the energy from high temperature, high pressure steam into rotational, mechanical energy, which in turn drives the gearbox. The gearbox reduces the speed of the turbine output shaft in the ratio of 10:1. The gearbox then transfers mechanical power to the DC Generator, and the DC Generator supplies DC voltage to a load resistor. The load resistor simulates a more useful electrical load

A. The 75 HP Steam Turbine Prime-Mover

The steam turbine operates at an approximate power output of 75 HP, maximum, at an output shaft speed of 12000 revolutions per minute (RPM). The horsepower (red) and torque in foot-pounds (green) characteristic curves, as a function of output shaft speed, are shown in Fig. 1.

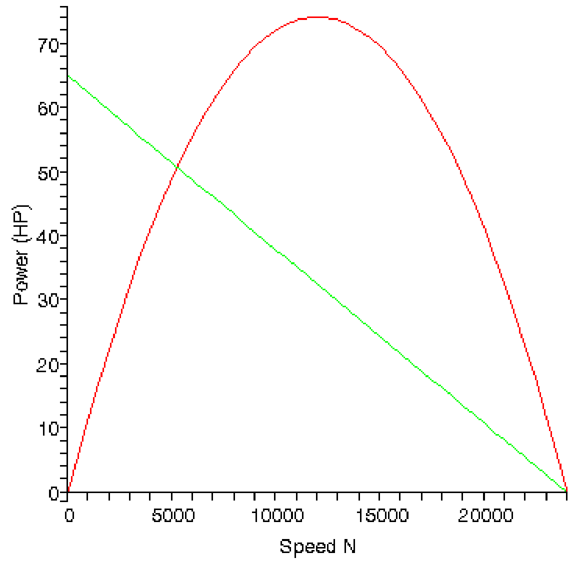

Figure 1. Steam turbine horsepower and torque versus speed.

These horsepower and torque characteristic curves were not measured from a real steam turbine – they are "representative" characteristics of a broad class of prime movers - e.g., gasoline or diesel engines, steam turbines, hydraulic turbines, windmills, gas turbines, etc. A characteristic common to prime movers is the nearly linear, downward sloping torque curve in the range of interest. In our hypothetical system the maximum available horsepower is available at approximately 12000 RPM and approximately 32.5 foot-pounds of torque. It is especially important to realize that the horsepower curve and the torque curve are not independent of one another. In such systems the horsepower output is the product of torque and speed.

For this analysis, we have idealized the torque curve to be linear over the range of interest. Had we used real world data, it would not materially affect our results, and a non-linear torque characteristic would unnecessarily complicate the math model. See Section XI - Appendix A for some real-world prime-mover data.

We should ask why the horsepower and torque curves are as they are. What if the torque curve were flat with no slope, or what if it has an upward slope instead of a downward slope? If the torque curve were constant instead of sloping downward, then the horsepower would be an ever-increasing function. There is no prime-mover known to man with such a characteristic. We are constraining the torque curve to be linear over our range of interest (+ or – 25% around the nominal operating speed of 12000 RPM). Since the torque curve is constrained to have a linear slope downward in the region of interest, it follows that there can only be one maxima for the horsepower curve, which is found by setting the first derivative of the horsepower with respect to speed to zero and solving for the speed..

B. *The 10:1 Gearbox*

A gearbox is often provided in order reduce the high shaft speed of the turbine to a speed more suitable for an electric generator. Generally speaking, steam turbines operate most efficiently at very high shaft speeds in comparison with DC generators. In our analysis we find the gearbox a convenient place to assign a 92% efficiency, which is intended to represent the total mechanical losses of the rotating components of the turbine and gearbox combined. Note that we do not model the steam source and conversion of steam energy to mechanical energy.

C. *The DC Generator*

A permanent-magnet, DC generator was chosen in order to avoid the complexities of self-excited series, shunt or compound generator configurations. The generator has been idealized to ensure a linear speed-voltage characteristic over the range of interest centered on 1200 RPM. The generator is sized for a power level of approximately 50 kW and is assigned a nominal efficiency of about 95%, which will be implemented as a series resistance of 1 ohm.

IV. A Mathematical Model for the System

The equations that will be used to model each component of the system will be given in detail, starting with the steam turbine. After the steam turbine, gearbox and DC Generator have been modeled, we will present a mathematical approach for determining the source impedance as seen from the load resistance looking toward the source. The math model models a power-limited system operating at its maximum available power output. The math model will be used to determine the operating point of the system in terms of the important parameters, such as speed, torque, horsepower, efficiency, electrical power generated and electrical power consumed in the load. Numerical examples will be given along with graphical plots in order to visualize the relationships among those variables.

The system output impedance will be measured using the model just as if we were measuring the values on a real system in the laboratory. Those measurements will be compared with more sophisticated mathematical analyses of the system. The behavior of the model in relation to the MPTT will be explored in detail – first to demonstrate and confirm the principles of the MPTT and also to contrast the interpretation and meaning of the MPTT with other theorems attributed to L. C. Thévenin and E. L. Norton. Contradictions will arise, and they will be dealt with appropriately.

A. *The Mathematical Model of the Steam Turbine*

As can be seen in Fig. 1, the steam turbine has a torque curve that is linear over the region of interest. The linear region has been extended to the intercept at speed = 0 in order to provide the linear expression for torque as a function of the torque coefficient in foot pounds per RPM, a stall torque in foot pounds, and the speed in RPM. The relationship between torque and speed is shown in (1), where T_S is the stall torque and k_T is the slope of the idealized torque curve.

$$T = T_S - k_T * N \tag{1}$$

Eq (1) shows that the torque starts out with a “stall torque” at zero RPM and decreases linearly to zero torque at some speed that is referred to as the “runaway” speed. A real-world prime-mover generally has a non-linear torque curve, but over the nominal operating range of interest, the torque vs. speed curve can be considered linear and characterized by the stall torque intercept at zero speed and the maximum runaway speed intercept at zero torque. Eventually, we will conclude that the slope of the torque curve in the linearized region of interest is analogous to electrical conductance and that its reciprocal is analogous to electrical resistance.

Since the steam turbine horsepower is the direct result of its delivered torque, we will rate the steam turbine as having an arbitrary stall torque intercept of 65 foot pounds and a runaway speed of 24000 RPM. Equation (2) defines a constant for converting the product of torque in foot-pounds and speed in RPM into horsepower (HP). Equation (3) is the relationship between power in units of horsepower (HP), torque T in foot-pounds and speed, N, in RPM.

$$k_N = 2\pi / 33000 = .0001904 \tag{2}$$

$$P = k_N * T * N \tag{3}$$

If we take the derivative of the power expression with respect to speed, set that equal to zero, and solve the resulting expression for speed, we will obtain the speed at which the developed horsepower is a maximum. The stall torque is 65 foot-pounds, as was stated earlier. The slope, k_T, of the torque curve is 0.002708333 as determined from the two intercepts, 65 foot-pounds and 24000 RPM. Performing the calculations for the speed at which maximum power is developed, we obtain 12000 RPM. Using that speed in (1), we obtain 32.5 foot-pounds of torque as the nominal operating point. Using that torque and the 12000 RPM speed we calculate the actual maximum available horsepower from (3), which is 74.25 HP.

B. *The Mathematical Model of the Gearbox*

The gearbox is extremely simple. It consists of an input shaft, driven by the steam turbine, and an output shaft, which drives the generator. In between those two shafts is a cluster of gears that reduce the speed by 10:1 and linearly amplify the torque by 100:1, while losing 8% of the energy. The math model for the gearbox output is simply the math model of the steam turbine modified by the 10:1 ratio and by the efficiency factor, which may be easily adjusted to any value. Therefore, from the steam turbine operational parameters, we can see that the gearbox would deliver 325 foot-pounds of torque at a speed of 1200 RPM, if it were 100% efficient. Fig. 2 shows a plot of the Torque in foot pounds (red) and the power output in horsepower (green) versus gearbox speed in RPM.

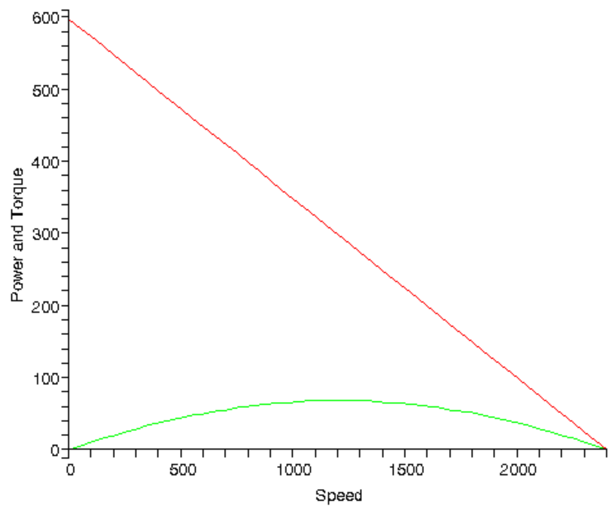

Figure 2. Gearbox power and torque versus speed.

At the nominal efficiency of 92%, it delivers only 299 foot-pounds of torque at 1200 RPM.

In order to model the gearbox output characteristics, we assume that it simply translates the speed, torque and horsepower characteristics of the steam turbine into the appropriate values at the 1200 RPM operating point with the power output reduced by 8%. The result is that we can linearize the output torque characteristic of the gearbox to follow the relationship in (4) and (5).

$$k_C = 746 \text{ (watts/horsepower)} * k_N = 0.14204 \quad (4)$$

$$P_{Gearbox} = k_C(T_S * N - k_T * N^2) \quad (5a)$$

The gearbox power output (5) gives the same relationship as (1) and (3), except that now the stall torque is 598 foot-pounds and the torque coefficient, k_T, is 0.24916 foot-pounds per RPM over the linear range from about 900 RPM to about 1500 RPM (our designated range of interest). Note that (4) is the definition of a new constant, k_C, which combines the earlier conversion factor with a new factor to convert horsepower to watts. The gearbox output power in watts at the maximum power operating point can now be determined as follows:

$$P_{Gearbox} = 0.14204\,(598 * 1200 — 0.24916 * 1200^2)$$
$$= 50965 \text{ watts} \quad (5b)$$

We express the power level in watts instead of horsepower because from this point forward all equations will express power in watts.

C. The Mathematical Model of the DC Generator

The math model for the DC generator combines a scale factor constant relating generated electromotive force (emf) to speed and a fixed armature resistance value of **one ohm** for $I^2 * R$ losses. If desired, this fixed value of "loss" resistance can be altered to include windage and friction losses. However, such losses are generally not as load dependent as are the copper losses. Our model provides only this one parameter for internal losses of the generator. The ratio of emf to speed is 0.83333 volts per RPM as shown in (6).

$$k_G = 1000 \text{ volts} / 1200 \text{ RPM} = 0.83333 \quad (6)$$

$$P_{Gen} = (k_G * N)^2 / R_{total} \quad (7)$$

$$R_{total} = R_L + R_i \quad (8)$$

Equation (7) gives the relationship between power generated by the generator, the emf constant, the generator shaft speed and the total circuit resistance R_{total}. These values result in a nominal, full power output of 1000 volts emf.

Fig. 3 is a plot of the total power generated by the generator in watts as a function of load resistance. Note that this is *total power generated* – not *total load power*. The load power is total power generated less the loss in the one ohm internal resistance.

Based on the full output power delivered to the generator from the gearbox, we can compute the nominal, full power load resistance based on (5b), (8) and Ohm's law as follows:

$$R_L = (emf^2 / P_{Gearbox}) - R_i$$
$$= (1000^2 / 50965 \text{ watts}) - 1.0 = 18.62 \text{ ohms}$$

Under these conditions, the generator generates 50,965 watts, of which 48,370 watts is delivered to the load and approximately 2590 watts is dissipated in internal losses (the 1 ohm armature resistance). The math model provides for adjustment of the internal loss resistance to any desired value, which indirectly allows us to control the generator efficiency through a fictitious series loss resistance to account for friction, windage, copper losses, etc. This is an approximation, because in a real generator some losses, such as the armature copper losses are proportional to the load and others are not. Windage losses, for example, are essentially constant.

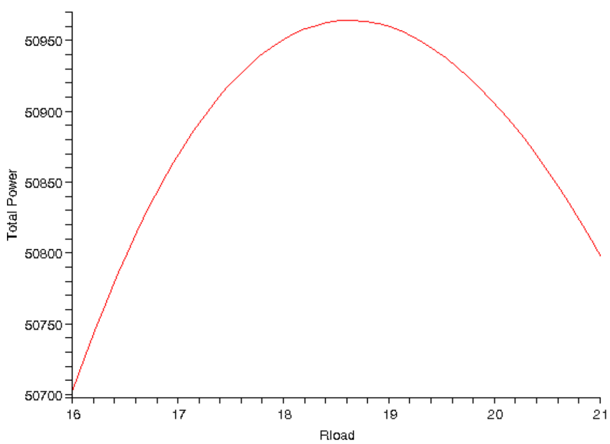

Figure 3. Total power generated (watts) as a function of the load.

As stated previously, the math model does not provide for separation of generator losses into different categories. This is justified on the basis that we are studying the behavior of this system over a very limited range of operating conditions, and the sum total of all losses can be approximated over that small range as being dependent upon load current only.

D. *The Mathematical Model of the Overall System*

The overall math model converts all mechanical power delivered to the generator from the steam turbine/gearbox elements, $P_{Gearbox}$, into total power generated by the generator P_{Gen} as shown in (9), including generator internal losses.

$$P_{Gen} = P_{Gearbox} \tag{9}$$

In other words, all power generated by the generator is simply a conversion from the mechanical power delivered from the gearbox and turbine into useful electrical power and a heat loss due to load-dependent losses. Setting total generated power equal to mechanical power out of the gearbox, (9), allows us to combine (7) and (8) with (5), from which we obtain 10.

$$N = T_S * k_C * (R_i + R_L) / (k_G^2 + k_T * k_C * (R_i + R_L)) \tag{10}$$

Before continuing, we should test our math model with some numerical values in order to see that it behaves rationally. To that end we solve for N using (10).

$$N = \frac{598 * 0.14204 * (1.0 + 18.62)}{0.83333^2 + (0.24916 * 0.14204 * (1.0 + 18.62)}$$

$$= 1200 \text{ RPM}$$

The solution for the speed as a function of total circuit resistance allows us to parametrically study the effect on output electrical power due to changes in the value of the load resistance, R_L, which is about 95% of the total circuit resistance, R_{total}. That, in turn, provides us with the ability to demonstrate the apparent source impedance in a conjugately matched, power-limited system that is delivering maximum output power to the load. Fig. 4 is a plot of the speed in RPM versus load resistance in ohms.

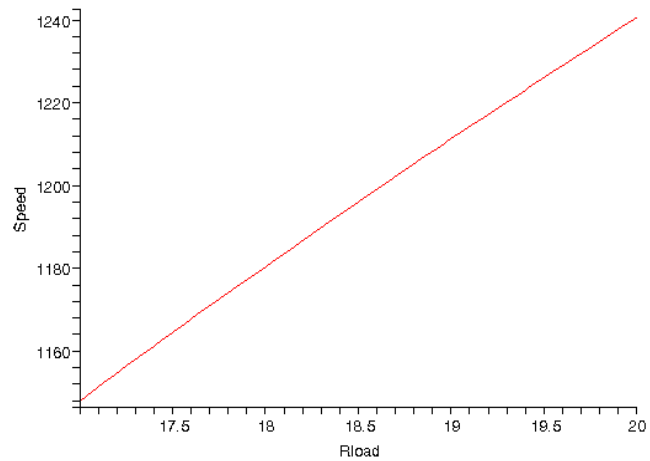

Figure 4. Speed versus load resistance of the overall system.

The nominal, full power operating point at 1200 RPM and 18.6 ohms load resistance is consistent with Fig. 4.

Note that had we chosen to include a gearbox efficiency parameter in the above equations, it would appear as a divisor of the k_G^2 term.

What exactly is meant by **power-limited,** and why is that important to our analysis? Also, what is meant by conjugate match, and is that important to our model? The next section is devoted to answering those questions.

V. How the Math Model Relates to the Maximum Power Transfer Theorem

The maximum power transfer theorem has at least three different areas of importance – the concept of the "conjugate match"; system efficiency; and a set of explicit and implicit constraints. The following subsections will address each of those areas individually.

A. *The Conjugate Match*

The concept of a "conjugate match" can be disposed of by reviewing electrical engineering theory. Two impedances that are complex conjugates of one another are two complex impedances that have equal real parts and equal but of opposite sign imaginary parts. For the case at hand, since we constrained our analysis to a DC circuit, the two complex impedances are simply two resistances which must be equal in order to be conjugately matched. If this analysis were to treat the more general AC case, the two impedances would be complex. In order to be complex conjugates, one impedance would be of the form a + jb and its complex conjugate would be a – jb. We will be concerned about the "apparent output impedance" of our generator in comparison with the load resistance under conditions of full power output.

B. Efficiency Concerns

We will model our system and analyze its behavior at the point of maximum power output, and we will be interested in confirming that the output impedance property of the system and the load on the system be conjugates of one another.

Our interest should be heightened in view of the overall efficiency of the system, which we know is in the neighborhood of 0.92 * 0.95 or approximately 87.4%. Why should there be a concern about overall efficiency of the system? An efficiency of 87.4% is a good efficiency, is it not?

Every EE student should recognize the problem. The classical interpretation of the MPTT and the Thévenin equivalent circuit model is that a circuit in which the source impedance and the load impedance are conjugates of one another dictates a maximum efficiency of 50%. If we claim that our system is conjugately matched and is also 87% efficient, there would be those who would argue that it is impossible. Nonetheless, we will demonstrate operation at the maximum available power output with a conjugate match between the load impedance and the apparent source impedance. The concerns about efficiency will be continued in Section V, subparagraph E.

C. Explicit Constraints of the MPTT

The two explicit constraints are those that together require that the load be varied while determining the point of maximum power transfer.

1) While Varying the Load

There is the explicit constraint in the MPTT that the maximum power will transfer to the receiving network "as the load impedance is varied". Interpretations and applications of the MPTT often ignore this explicit constraint or fail to mention it. It is quite a different situation to design a source to drive a fixed load efficiently. That would require that the source resistance be kept as near zero as possible.

2) Maximum Power Will Be Absorbed…

"Maximum power" is the maximum power that the system is capable of – not the maximum power that it is capable of delivering efficiently and not the maximum power that it is capable of delivering without self destructing. In order to determine the maximum power that a system can deliver, one must know the power as a function of the load impedance and find its maximum. This arises directly from the constraint, "while varying the load". The derivative of the power as a function of the load impedance must be set to zero and solved for that unique value of load impedance that will cause a maximum transfer of power. This is an explicit constraint in the MPTT.

D. Implicit Constraints of the MPTT

One has to ask, how could there be a "maximum power", unless the power is somehow limited? If the power capability of a system were unlimited, then how could we define its "maximum" power? A "power-limited" system must be a system that has a well-defined maximum power output – a characteristic that is actually common to every system built by man. A system which is not power-limited would be a system that would continue to provide more and more power, indefinitely, no matter how much the load is increased.

The reason that we call attention to the implied constraint that the system must be power-limited is that we intend to conduct an analysis at the point of maximum power output, and we want to emphasize that not only is it a practical consideration that the system be power-limited, but also that it must be operating at exactly the level to which it is power-limited. That will become a crucial part of our understanding of that property of the system called the "apparent output impedance" or "apparent source impedance".

This would be a good time to review the maximum power transfer theorem and reflect on the words, "*maximum power* will be absorbed …". The maximum power transfer theorem obviously applies *only* to systems that are both *power-limited* and *operating at maximum power*. The words cannot be interpreted any other way.

For the purposes of this article and our analyses, we will be even more specific in our interpretation of both **power-limited** and **maximum power**. The term **power-limited** will be understood to mean the power is limited **with respect to a varying load**, and the term **maximum power** will be understood to mean the maximum **with respect to a varying load**.

We conclude that the maximum power transfer theorem explicitly requires a power-limited system that is operating at its maximum available power.

E. Introduction of the Non-dissipative Resistance

If the source impedance and the load impedance are to be complex conjugates of one another, then according to the usual interpretation of Thévenin's theorem, the efficiency cannot exceed 50%. The reason for the apparent contradiction between Thévenin's theorem and the MPTT is that neither the MPTT nor Thévenin's equivalent circuit explicitly recognizes that the power source can exhibit an "apparent" impedance that is much larger than the loss elements or physical source impedances of the system. Those theorems also did not envision an “apparent” impedance that does not convert energy into heat or consume the energy as would a conventional, physical impedance or loss element. The following extract gives us some historical insight into the conventional wisdom at the time Professor Jacobi introduced his theorem of maximum power transfer:

> *Professor Moritz von Jacobi of St. Petersburg (1801-1874) is probably often confused with his younger brother Carl, the eminent mathematician of Königsberg (1804-1851)…… Jacobi quite correctly concluded that electric motors were uneconomic, considering the high price of zinc and the 50% loss of energy. The concept of energy was as yet somewhat hazy, and the fact that mechanical work out was equal to the electrical work done against a counter-emf was unknown, at the time. However, it was adopted as a maxim that the internal resistance equaled the load resistance for maximum power……If R' = 0, there is no external power. If R' =∞, there is also no power, since I = 0. Therefore, for some intermediate value of R' there must be a maximum power. Calculus gives the result easily, but a little reasoning also shows that maximum power is attained when R' = R (imagine interchanging R and R'). Hence the theorem: Maximum power is transferred when the internal resistance of the source equals the resistance of the load. We should carefully note the condition that is seldom added: When the external resistance can be varied, and the internal resistance is constant. ……When Edison was designing his lighting system in 1880, the received wisdom was to make the armature resistance equal to the resistance of the load. Either he, or Upton, his mathematical advisor, saw that this was quite incorrect. The Z dynamos and the Jumbos were made with very low armature resistance, and at one step he obtained efficiencies of 90%. He was ridiculed in the technical press by American "experts" who proved conclusively that he could not have done what he in fact did. Edison's inefficient field structures increased the weight of the dynamos, but did not affect their electrical efficiency. [2]*

The wording of the MPTT is very explicit about the load resistance being varied. Dynamo designers were working the other way around. They were assuming a specific load and trying to design the generator for an “optimum” source resistance. They took the wisdom of the day literally. They assumed that 50% would be the ultimate efficiency, but Edison (or Upton) figured it out. What no one was familiar with is the concept of dynamic, non-dissipative source impedance. Unfortunately, many people to this day are equally confused. We will demonstrate the existence of dynamic, non-dissipative resistance as part of our analysis of the system.

VI. Measuring the “Apparent Source Impedance”

If we wanted to model the problem of measuring the “apparent source impedance”, we would undoubtedly start with either a Thévenin equivalent circuit or a Norton equivalent circuit. The two types are interchangeable and the analyst can choose whichever is more convenient. We will look at both methods. Unfortunately, the Thévenin equivalent circuit, as fundamental as it is in circuit analysis, is the very reason for the general lack of understanding of the “apparent source impedance” and its relationship to the MPTT. We will explain why after reviewing the relevant equations.

A. The Thévenin Equivalent Circuit

The Thévenin equivalent circuit is a constant emf source (such as an ideal battery), and an internal source impedance and a load impedance, all in series. What we want to do is to simulate the indirect measurement of the internal source impedance of our system by means of our math model and its Thévenin equivalent circuit. It is presumed that for one reason or another, we cannot use an ohmmeter or external source of power to measure the internal source impedance directly. The simplest approach would be to short the output and measure i_{SC}, and to open the output and measure E_{OC}. The internal source impedance is then found from E_{OC} / i_{SC}. For many systems such an approach would be dangerous. A preferred approach is to measure the current flow in two different, accurately known load resistances and apply that information to determine the internal source impedance by means of the well-known relationship (11).

$$R_{Source} = (i_2 * R_2 - i_1 * R_1) / (i_1 - i_2) \quad (11)$$

The advantages of this method are that it is not intrusive, and the measurements can be made at or near the normal operating point, thereby avoiding dangerous excursions beyond the normal operating region.

B. The Norton Equivalent Circuit

The Norton equivalent circuit is a constant current source, a shunt source conductance and a shunt load conductance. Again, the problem is to indirectly measure the internal source impedance envisioned as a Norton equivalent circuit, and again, we cannot measure the internal source impedance directly. The simplest approach to measuring the internal source impedance is to disconnect the load and measure the open circuit voltage E_{OC} and to short the output and measure the i_{SC}, and again, the internal source impedance is found from E_{OC} / i_{SC}. Once more, for our system and its math model it is better to measure the voltage across two different, accurately known load resistances, one after the other, and apply that information to determine the internal source impedance using (11). Note that the same equation applies, regardless of whether you envision it as the Thévenin or the Norton equivalent circuit.

C. The Issue of Linearity

Unfortunately, our system equation for speed as a function of load resistance (10) is not linear over the range 0 to infinity ohms. Whereas the Thévenin theorem assumes a constant emf, as from a battery, in our system the emf is dependent upon the load, because the load affects the turbine's equilibrium speed, which in turn affects the generated voltage.

We will proceed with measurements as if our system were enclosed in a black box and will determine the effect on the parameters of the system as a result of load changes, even though we cannot directly relate the results to the traditional interpretation of the Thévenin theorem and the Thévenin equivalent circuit, although it should be noted that the equation is quite linear over the limited range of interest around the full load operating point, as can be seen in Fig. 4.

D. *Measuring in the Abstract*

We will substitute values into (10) for two separate values of load resistance, R_L. One of the two load resistances will be that of the nominal full power load, and the other resistance will be for a 5% higher value than the nominal.

There will be two values for the speed N in RPM using (10), which gives the speed as a function of the resistive load. For each value of N we will calculate the corresponding emf generated by the generator using the coefficient k_G.

Knowing the voltage for each case and the total circuit resistance for each case (load resistance + 1 ohm loss resistance), we calculate a current flow for each, based on conventional circuit theory and the reasonable assumption that the solution will be an "apparent" source resistance, based upon the "black box" behavior.

The following table of data represents the two data points needed. (Remember that the data for the first point is simply the theoretical, full power operating point already calculated for the system.) The second point is another theoretical point using a load resistor 5% higher than the first resistor. The new speed, emf and current flow then follow directly from the equations in our math model and Ohm's law.

TABLE I. LOAD DATA

	R_{load}	N	emf	I
Nominal load resistance	18.62	1200	1000	50.96
5% Higher load resistance	19.55	1228	1023	49.78

Inserting the above values into (11):

$$R_{Source} = \frac{49.78 * 19.55 - 50.96 * 18.62}{50.96 - 49.78} = 20.6 \ ohms$$

We have used (11) to calculate a source resistance based on a Thévenin equivalent circuit, even though we acknowledge that the conditions for the Thévenin theorem have not been met, for the simple reason that (11) is still an appropriate equation for determination of an "apparent" source resistance of a "black box" system.

The 20.6 ohms apparent source resistance will be obtained for any reasonable change in load resistance and demonstrates clearly that the apparent total source impedance is 20.6 ohms. Note that this value is *inconsistent* with the known 1.0 ohm internal resistance in our math model. It is also *inconsistent* with the known efficiency of the system (87.4%). We have encountered what appears to be a resistance of approximately 20.6 – 1 = 19.6 ohms, which we know does not exist as a physical, dissipative resistor in our system model.

As indicated in Section V, subparagraph D, this is an example in which the Thévenin equivalent circuit and the MPTT seem to disagree. How can the system be conjugately matched and also exhibit characteristics of 87% efficiency? The answer lies in the fact that neither the MPTT nor the Thévenin equivalent circuit models provide for a dynamic, non-dissipative resistance. In other words the source impedance "appears" to be a resistor that would dissipate several thousand watts, but in fact is not physically present as a resistor and does not dissipate any energy. Obviously, we need additional understanding of the true nature of the "apparent source impedance" that we have just measured in the abstract.

The problem lies in the fact that neither the MPTT nor the Thévenin (or Norton) equivalent circuit model can explicitly account for anything beyond a very simple circuit or network consisting of pure circuit elements, such as resistors, capacitors, and inductors.

In our math model, we were able to model the physical behavior of the prime-mover, which is a separate and independent sub-system, capable of modifying its operating point as the load changes. Our model incorporates both **power-limiting** and an operating point of **maximum power output**. Those two characteristics model a behavior that cannot be modeled or predicted using the Thévenin equivalent circuit, and that is why we indicated in Section V that dependency on the Thévenin equivalent circuit is the root problem.

VII. ADDITIONAL INTERPRETATION OF THE RESULTS

Calculating the "apparent source impedance" from (11) gave a result of 20.6 ohms. One might expect that looking back into the generator, one would see the internal resistance of 1 ohm as the source impedance. That's the value in our math model and that's the value one would expect to measure based on a Thévenin equivalent circuit model. It fits our traditional concept of "source resistance".

What other insight can we get from our measurement in the abstract?

A. *Looking at the Data Itself*

Forgetting about the measured source resistance for a moment, what do we see from a simple inspection of the data itself? The above table of results indicates that the nominal, full power load resistor is 18.62 ohms, and that under those conditions the generator speed is 1200 RPM, the generated emf is 1000 volts and the current flow is 50.96 amps (50960 watts of electrical power generated, including the generator's internal losses). Likewise, the table shows clearly that when the load resistor is increased in value by 5%, the system operating point changes to a speed of 1228 RPM and correspondingly generates an emf of 1023 volts with a current flow of 49.78 amps (50924 watts of electrical power generated, including the generator's internal losses – slightly less power than above). Let's work back through the system and see if we can develop some insight as to why we get this result. First, is it reasonable that changing the load resistance upwards should have the effect of increasing the speed of the prime-mover? The immediate effect that one would expect from increasing a load resistance would be that the current in the load would drop by a similar percentage – i.e., about 5%. (We actually see a drop of only 2.3%).

We would also expect to see a reduction in generated electrical power (it reduced by approximately 0.06%). We would also expect the steam turbine speed to *increase*, since its shaft is connected through the gearbox to the generator. Let's look again at why the speed increased. A smaller electrical load on the generator means the driving shaft has less reaction torque – i.e., the reduced electrical load reduces the back torque upon the gearbox, and that allows the shaft speed to accelerate. In other words, the lighter load on the generator immediately causes an imbalance between the applied torque provided from the gearbox and the reaction or back torque produced by the work being done (mechanical energy being converted to electrical energy). The exact same effect occurs between the gear box and the turbine, since all three components are rigidly connected together. It is reasonable to expect an electrical load change to reflect the force imbalance directly back to the gearbox and thence back to the turbine.

In summary, a reduced electrical load on the generator causes a force imbalance in the directly connected drive shafts of the system. That imbalance causes all three components to accelerate to a new, higher equilibrium speed, 2.5% higher than the original speed. As a direct consequence of the torque and horsepower curves of the turbine, the shaft speed automatically reaches a new equilibrium between driving torque and reaction torque. Why was the resulting speed change only 2.25%, when the load resistance change causing these reactions was 5%? Is this readjustment of the torque/horsepower operating point of the power-limited prime mover in some way responsible for the "apparent" source impedance?

B. Operating Point Considerations

Is this result consistent with such parameters as the torque coefficient – i.e., the slope of the torque curve for the generator or the gearbox? The system was operating at the full power point of the torque/horsepower curves. Suddenly, the reactive torque from the generator is reduced, and it now requires a smaller driving torque to maintain a given speed. The result is that the operating point moves to the right in the torque/horsepower curves (Fig. 1), instead of to the left, because that is the only region consistent with a lighter load requiring less torque and higher speed. The new operating point is consistent with a smaller electrical load caused by a higher load impedance.

As pointed out above, the operating point on the horsepower curve shifts to the right. Of course, that has to happen if the speed has increased, but is the new operating horsepower level consistent with a lighter load? The original operating point was at the maximum possible horsepower – full power. A speed change to either the right or the left from the point of maximum power output in Fig. 2 is a point of less power output from the prime mover (steam turbine and gearbox).

C. Why Did the Speed Change Only 2.5%?

The remaining question is, why is the new operating point only ***2.25%*** *higher* in speed when the load has *decreased by* ***5%***? Actually, we were getting ahead of ourselves when we thought that a load **resistance** increase of 5% would result in a load **current** drop of 5%. We see that it actually dropped by only 2.3% (from 50.96 to 49.78 amperes). Why is it that even though we thought we were changing the load by 5%, the system reacted by finding a new equilibrium operating point 0.0625% lower in power instead of 0.25% lower in power? Therein lies the secret to an understanding of "apparent source impedance".

In a simple circuit with a fixed emf and a source impedance equal to the load impedance, a 5% *increase* in the load resistance would *decrease* the current in the load by only 2.5%. *Our system behaves as if it were conjugately matched!* How is that possible, since the known internal loss resistance of the system is only 1 ohm – not the 20.6 ohms that we are measuring? We have encountered a non-dissipative, "*apparent*" source resistance that is created by the inherent action/reaction characteristics of a power-limited system.

D. The Non-Dissipative, Apparent Source Resistance

In spite of the fact that this numerical analysis has clearly shown that our system is conjugately matched and is obeying the Maximum Power Transfer Theorem, it is difficult to grasp that it is also operating at approximately 87.4% efficiency (92% gearbox efficiency, and 95% generator efficiency) and also has an "apparent source impedance" equal to the load impedance.

If there is any one single issue that is the crux of this phenomenon, it is the fact that the prime-mover is **power-limited** and **operating at its maximum available power output**. Those characteristics, which in this case are the direct result of its power curve having a single maximum and a fall off of power at either side of that maximum, cause our system to react to load variations in such a way as to produce the same effect as a simple constant emf and conjugate matched circuit. The power limited system operating at its maximum power output reacts in a way that a passive circuit element or passive network cannot react. It is inherently able to automatically adjusts its power output to a lower value whenever the load impedance deviates from the conjugate match condition. The effect of that change is that the system creates a response exactly as if there were a conjugate impedance match.

As our analysis continues, we will demonstrate that the *power-limited system, operating in a maximum power output condition* will exhibit an apparent source impedance, which automatically provides a conjugate match to a load accepting that power output. What else can be said about the value of that apparent source impedance? Is the particular load that can be matched determined by the system or will the system match any load presented to it? The system is incapable of matching all loads, because the maximum power transfer theorem is fundamentally tied to the concept of a power-limited source. It is only when the maximum available power is being delivered that the system automatically provides the conjugate match. This is not to say that there is only one possible value of source impedance for the system. It is dependent upon the system parameters, such as the prime mover's torque/horsepower curves and the speed/voltage characteristic of the generator. However, once those characteristics have been established and the power available to the primer mover is determined, the apparent output impedance is determined.

VIII. The Derivative Analysis

Consider a simple series DC circuit consisting of an emf, a source resistance and a load resistance. Let a small positive change in the normalized load resistance be recorded as ΔR / R and let the resulting change in normalized current be recorded as $\Delta I / I$. The ratio $\Delta I*R / \Delta R*I$ should equal –0.5 when there is a conjugate match. That is, the *increase* in resistance should cause a *reduction* of current – thus, the *negative sign*. For example, with 10 volt source voltage, 5 ohm source resistance and 5 ohm load resistance and a 1% increase in load resistance, the normalized load resistance change of 1% and the normalized current change of -0.5% would give a ratio of –0.5

This is the logical consequence of the fact that exactly half of the total circuit resistance has changed by the percentage 100 * $\Delta R/R$. If the entire circuit resistance had been changed by 100 * $\Delta R/R$ percent, then the current would also change by the same percentage, and ΔI / ΔR would equal –1.0.

To continue with our analysis using the derivative, we take the derivative of the load current, I, in our system with respect to the load resistance R_L. If our system is to give the appearance of being conjugately matched, this should result in –0.5 when properly normalized with the nominal values, R_{L0} and I_0.

Equation (12b), below, is the equation for the derivative of current with respect to the load resistance R_L. The equation for current (12a) was obtained by multiplying the speed from (10) by the factor k_G / R_{total}, which is the conversion from speed to current at a given power level. Note also that in taking the derivative with respect to R_L, the 1 ohm constant internal loss term disappears from the R_{total} term, leaving only R_L.

$$I = \frac{T_S * k_C * (R_i + R_L)}{(k_G^2 + k_T * k_C * (R_i + R_L))} \left\{ \frac{k_G}{R_i + R_L} \right\} \qquad (12a)$$

$$\frac{dI}{dR_L} = -\frac{T_S * k_C^2 * k_T * k_G}{(k_G^2 + (k_T * k_C * R_{total}))^2} \qquad (12b)$$

We then apply a normalization factor as follows:

$$\left.\frac{dI}{dR_L}\right|_{Norm} = -\frac{T_S * k_C^2 * k_T * k_G}{(k_G^2 + (k_T * k_C * R_{total}))^2} \left(\frac{R_{L0}}{I_0} \right) \qquad (13)$$

The normalization factor normalizes the amperes per ohm derivative to a dimensionless derivative. We now compute a numerical value for the derivative using the same system coefficients and parameters used in the preceding sections.

$$\left.\frac{dI}{dR_L}\right|_{Norm} = -\frac{598 * 0.14204^2 * 0.24916 * 0.83333}{((0.83333)^2 + (0.24916 * 0.14204 * 19.62))^2} \left(\frac{18.62}{50.967} \right) = -0.474 \qquad (14)$$

The small discrepancy between the predicted value of –0.500 and our value of –0.474 is due to the *one ohm* of loss resistance that we incorporated in the generator for realism. When we examine the following plot, we will see that the conjugate match exists when the load resistance is about 20.4 ohms, which is more consistent with our measurement in the abstract in Section VI, subparagraph C.

Fig. 5 is a plot of the normalized derivative of the current plotted versus the load resistance. It is consistent with (14) and shows that a conjugate match actually exists at about 20.4 ohms load resistance.

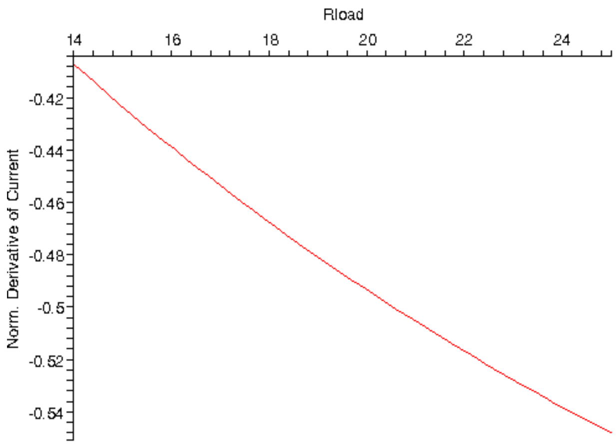

Figure 5. Normalized derivative of current versus load resistance.

If all equations are re-computed, starting with (8), using zero internal resistance for the generator, we obtain a nominal, full power load resistance of 19.62 ohms instead of 18.62 ohms, and when we calculate the normalized derivative of current with respect to load resistance plotted versus the load resistance, we obtain the results shown in Fig. 6, which show that a conjugate match exists at a load impedance of 19.62 ohms. It can be shown that as the generator internal losses are increased, arbitrarily, the conjugate match will still occur when the internal and load resistances total approximately 19 ohms.

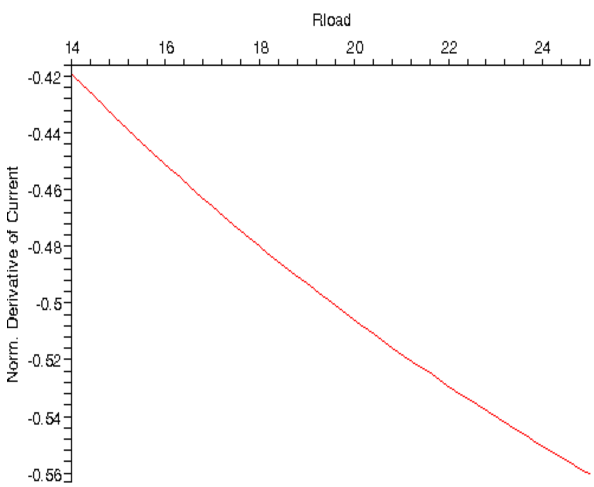

Figure 6. Modified for internal resistance of generator = 0.

IX. Summary of the Analysis of the Steam Turbine, Electrical Generator System

Let us again review the relationships that have been demonstrated in this system. First we conducted a numerical analysis of the system which ultimately determined that the system exhibits the phenomenon of an "apparent source impedance" that creates a conjugate match with the load in concert with the Maximum Power Transfer Theorem.

Second, we conducted an informal trace of cause and effect through the system in which the action of a disturbance of the load resistance causes a series of actions and reactions throughout the system that ultimately re-position the operating point of all elements of the system. The reactions are simply the result of an acceleration or deceleration of the rotational masses due to an imbalance of forces that persists until the forces return to equilibrium at a new operating point. In other words, the physical laws of action and reaction take over and force the system to reach equilibrium at a new operating point.

Third, we conducted an analysis using the derivative of the current with respect to the load resistance, which shows in yet another, independent way that the system is conjugately matched while maintaining the high efficiency expected of this system. The derivative method allows us to close the loop mathematically, so that we get a closed form solution to the question as to where the new operating point converges after the load change.

Fourth, we can say with confidence that the steam turbine and gearbox are insensitive to the internal workings of the electrical generator – the generator causes the same effect on the prime-mover/gearbox as would a friction brake insofar as loading is concerned. If a friction brake load were to be perturbed around the nominal operating load point, the behavior would be the same. That is, a 5% *increase* in the friction load would cause a shift of the operating point to a new operating speed 2.5% *lower* than the nominal full power operating point (and *vice versa*)– demonstrating that it is conjugately matched and yet operating at much higher than 50% efficiency. That is because the mechanical system is insensitive to the fact that the generator is converting the energy to electrical energy – the generator is just a mechanical load on the gearbox. The "apparent source impedance" concept is not limited to electrical circuits – it also manifests itself in mechanical systems, as will be demonstrated in Section X.

In summary, we recognize that the MPTT requires that the system be power-limited and be operating at its maximum power level in order to exhibit the characteristics of a conjugate match. We also claim that the theorem is reciprocal. That is, if the system is known to be conjugately matched, then we can conclude that it will demonstrate the characteristics of a power-limited system operating at its maximum available power output.

It should be apparent that the MPTT with its included concept of a conjugate match between source and load is not restricted to radio transmission lines and antennas, or even to electrical circuits. The MPTT should apply to any system in which there is energy flowing, without regard to the form of energy or the conversions between different forms of energy that may take place in the overall path of energy flow.

Our analysis has revealed an explanation for the contradictions that seem to exist between our MPTT-based system model and the Thévenin equivalent circuit of a matched load. Our explanation of these contradictions is based on the ability of the **power-limited system operating at its maximum power output to adjust its power output to a lower value, without incurring or imposing any additional losses, whenever the load impedance deviates from the conjugate match condition**. The system has created a response exactly as if there were a source impedance conjugately matching the load impedance. Conversely, a simple Thévenin or Norton equivalent circuit is incapable of such action and cannot exceed 50% efficiency.

Our analysis has suggested a mechanism described as a **non-dissipative** source resistance or **apparent source impedance** that is a unique feature of the power-limited system operating at its maximum available output power level. Conversely, such a mechanism is not recognized in the Norton or Thévenin equivalent circuits. At the time those theorems and the MPTT were developed there was no recognition of a dynamic or apparent source impedance that did not dissipate energy. The following section will continue with the further development of that concept and will demonstrate that the apparent source impedance is also a constant value for a given system that can be determined from that system's attributes.

X. Analysis of an All-Mechanical, Conjugately Matched System

For this analysis we will remove the electric generator of the previous analysis and install in its place a viscous damper load. The objective of this analysis is to extend the concept of the "apparent source impedance" to an all-mechanical system in order to demonstrate that it is applicable to purely mechanical systems as well as to electric, electro-mechanical, or electronic circuits.

A viscous damper load or rotational damper/dashpot is the mechanical analog of the electrical load resistance that we used to load the system in the previous sections. Other devices would do equally well. For example, the "prony brake" is an adjustable friction load for rotational systems. Other examples might include a ship's propeller, a ceiling fan, or a hoist that is continuously lifting a specified weight at a specified number of feet per second.

A. The Viscous Damper

The viscous damper, which is the mechanical analog of electrical conductance, is given the symbol *B*. If we define *B* as shown in (15), then it will have the units of foot-pound-minutes. Later, we will introduce another term, the inverse of *B*, and we will refer to it as "mechanical resistance".

$$B = T / N \tag{15}$$

B. What Happens as the Viscous Damper Load Changes?

Our next objective is to determine how the system will behave when the friction load is varied around the nominal operating point. We should be able to determine that the system behaves as a conjugately matched system. I.e., the friction load should be a conjugate match to the "apparent source impedance".

Following the same logic as was used in the derivative analysis, we consider what happens if the load *increases* (increase the value of *B*). The expected effect would be for the reaction torque or back torque of the viscous damper to increase, causing the output speed to *decelerate* until an equilibrium is reached. The situation we are describing here is analogous to putting on the brakes of an automobile. The addition of brake resistance causes a back torque or reaction torque that immediately slows the rotation of the braking wheels. I.e., the back torque exceeds the driving torque resulting in a deceleration of the rotational speed.

We already know from our analysis with the DC generator than when the load is decreased, the resulting decrease of back torque is reflected back into the gearbox and ultimately back into the steam turbine causing an immediate acceleration, and the system settles to a new operating point at a higher speed *and at a lower power output*. The question at hand is - what is the new operating point and how big is the percentage change in delivered load torque in relation to the percentage change in the viscous damper value *B*?

C. Power from the Gearbox and Power Dissipated in the Load – Choosing a Load Value

The equation for the output horsepower being absorbed in a viscous damper load attached to the gearbox at full power output is shown in (16) (combining (3) and (15)). The difference between this expression and the expression in (3) is only that (16) is a function of speed and the damper coefficient, whereas (3) gave horsepower as a function of the speed and torque produced in the prime-mover. Equation (15) has allowed us to eliminate the torque term from the equation. Think of the power in (16) as the power "consumed" or "absorbed" in the load and expressed as a function of the load device itself, whereas the power developed by the prime mover is a function of its torque-speed curve and is the power developed by delivering a given torque at a given speed.

$$P = k_N * N^2 * B \tag{16}$$

If the units for *B*, the rotational damper value, are chosen to be in pound-feet-minutes, and if the power is expressed in horsepower, then the value of *B* needed to dissipate the nominal full power output of the system is as shown in (17) (remembering that the full power operating point was 68.31 HP at 1200 RPM).

$$B = 68.31 \text{ HP} / ((1200 \text{ RPM})^2 * 0.0001904)$$
$$= 0.24916 \text{ foot-pound-minutes} \quad (17)$$

In one sense, this damper coefficient might result in confusion, because it is numerically the same value as the torque coefficient of the prime mover. This is the coincidental result of making the torque curves linear. A more complex torque curve for either the primer-mover or the viscous damper is possible, so what if we were to use some other value for *B*?

The constant, k_N, is the same conversion factor between speed in RPM, torque in foot-pounds and power in units of horsepower, as was defined in (2).

We might also take notice at this time that the torque coefficient in our math model for the turbine and gearbox is also analogous to electric conductance, since torque is the “through” variable (the analog of current) and rotational speed is the “across” variable (the analog of voltage). In that case the “mechanical” conductance analog is a source conductance instead of a load conductance.

The electrical power developed at the output of the gearbox at full power was shown in (5). We now modify that equation to give the power in horsepower as shown in (18).

$$P_{Gearbox} = k_N\,(T_S * N - k_T * N^2) \quad (18)$$

This gives us an equation for the HP developed at the output of the gearbox as a function of speed - the same as (5) except for the watts per horsepower conversion factor. The stall torque and the torque coefficient, k_T, are as developed in the previous analysis – namely, 598 foot pounds and 0.24916 foot-pounds per RPM, respectively.

D. *The Two Power Curves*

It would be helpful at this point to examine the two expressions for power, (16) and (18) versus speed. The horsepower output developed by the gearbox (green) and the horsepower absorbed by the viscous damper load (red) are plotted versus *N* (RPM) in Fig. 7.

In the same way that we set the generator’s generated power equal to the power output of the gearbox, we now set the viscous damper’s dissipated power (16) equal to the power output of the gearbox, (18). The full power operating point at 1200 RPM is clearly indicated in Fig. 7 as the intersection of the two power curves.

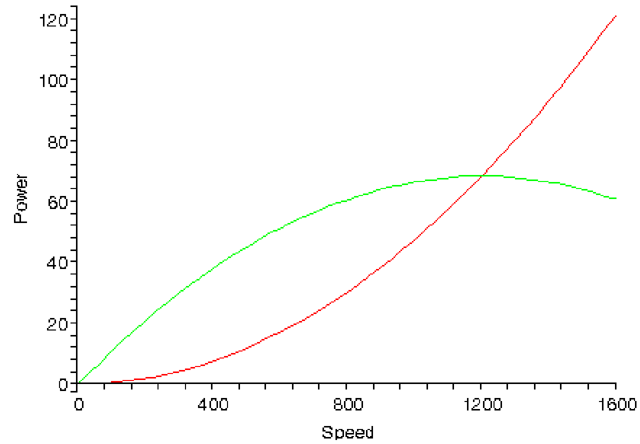

Figure 7. Horsepower versus speed.

E. *System Speed as a Function of the Viscous Load*

After re-arranging to solve for the speed, we obtain (19), which is an expression for the system speed as a function of the viscous damper value, *B*.

$$N = T_S / (k_T + B) \quad (19)$$

Remembering that k_T is 0.24916 from our math model of the gearbox, and also noting that our viscous damper value is 0.24916 from (17), we can check our equations by computing the nominal full power speed as shown in (20).

$$598 / (0.24916 + 0.24916) = 1200 \text{ RPM} \quad (20)$$

The result of 1200 RPM validates our equations to this point and shows that at the nominal full power output of the system, everything is in balance. That is, the horsepower generated at the gearbox output is absorbed by the viscous damper load. The system is in equilibrium (shaft speed is constant) at this operating point. We can also see the expression for speed in RPM versus damper coefficient, (19), plotted in Fig. 8.

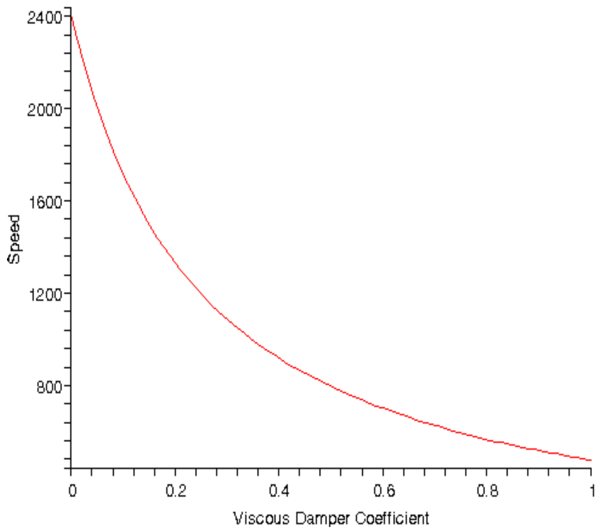

Figure 8. Speed versus damping coefficient.

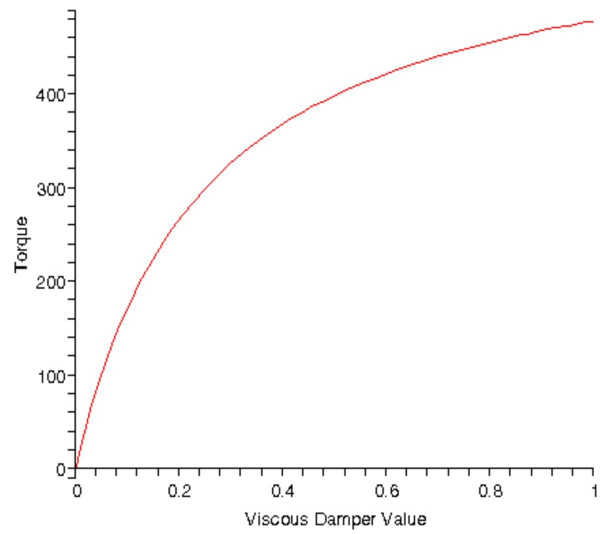

Figure 9. Torque versus value of viscous damper.

F. Torque as a Function of Viscous Load Coefficient (B) - (Analogous to Current as a function of Load Conductance)

It will be helpful for us to have an expression for the torque as a function of the viscous damper coefficient. Combining (1) with (19) to eliminate the shaft speed, N, we have an equation for Torque as shown in (21).

$$T = k_T * T_S * \left(\frac{1}{k_T} - \frac{1}{k_T + B} \right) \tag{21}$$

The torque versus viscous damper value is plotted in Fig. 9. The nominal, full power torque of 299 foot-pounds occurs at the nominal value of 0.24916 for the viscous damper.

All that remains is to take the derivative of the output torque with respect to B in order to find out whether or not the system is conjugately matched. However, in order to better visualize the parallels between this all-mechanical system and the turbo-electric system we will use the reciprocal of B.

G. Torque as a Function of Mechanical Resistance (Analogous to Current as a function of Load Resistance)

We will call our new term "mechanical resistance" and give it the symbol R_m. Equation (22) is just (21) re-written with $1 / R_m$ instead of B.

$$T = k_T * T_S * \left(\frac{1}{k_T} - \frac{1}{k_T + (1/R_m)} \right) \tag{22}$$

Fig. 10 is a plot of T versus R_m. Remember that our nominal value of B is 0.24916. Therefore, the nominal value of the mechanical resistance will be approximately 4.0135. As can be seen in Fig. 10, when R_m = 4.0135 the torque is 299 foot-pounds, which is the nominal full power torque.

The shape of the curve in Fig. 10 is worth noting. It has a shape similar to the curve of current versus load resistance in a Thévenin equivalent circuit (See Fig. 18 in Section XII - Appendix B). That is, the torque (or current) at short circuit load (runaway speed) starts out at the stall torque (short circuit current) and reduces rapidly as the impedance increases.

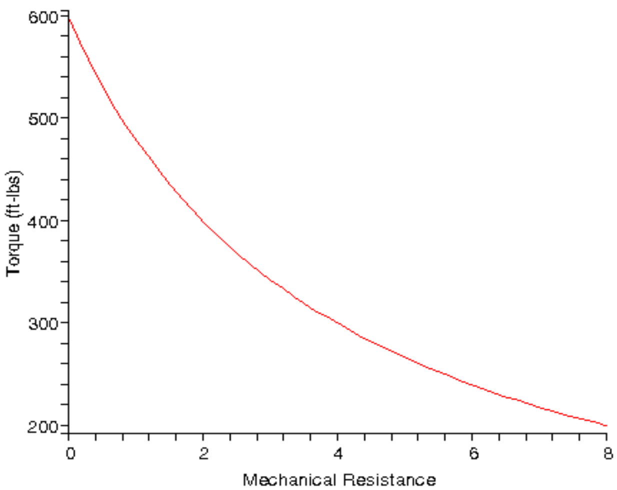

Figure 10. Torque versus mechanical resistance, R_m.

As we approach the point of conjugate match, the rate of change of torque (current) with respect to load impedance is reduced. As we go past the conjugate match point and continue to increase the load impedance, torque is still decreasing but at yet a slower rate of change with respect to load changes. The torque continues to reduce until at high impedances the torque (current) has become asymptotic to the runaway (zero) torque (zero current) value. Remember that the Thévenin equivalent circuit in our model is a fixed source impedance and a varying load impedance. The fact that these two curves have almost identical shape suggests that our all-mechanical system has a constant, non-zero source impedance or output impedance just as in the Thévenin equivalent circuit.

This prompts us to take a closer look at (22), which is re-arranged here as (22a).

$$T = \frac{T_S / R_m}{k_T + (1/R_m)} \tag{22a}$$

Equation (22a) is of the same form as the equation for load current as a function of load resistance in a Thévenin equivalent circuit. Therefore, our system can be characterized as having a constant, non-zero source impedance just like the Thévenin equivalent circuit. Is this true of all power-limited systems? Our system is representative of all power-limited systems operating at their maximum power level, and for all practical purposes, all systems are power-limited systems. Systems that are not operating at their maximum available power level, however, would be excluded.

H. Another Derivative Analysis

Taking the derivative of (22) yields (23). Note that we have also normalized the derivative using the nominal, full load values of B_0 (0.24916) and T_0 (299 foot-pounds). (*1/Bo* normalizes R_m).

$$\left.\frac{dT}{dR_m}\right|_{Norm} = -\frac{k_T * T_S}{(k_T + 1/R_m)^2 * R_m^2} * \left(\frac{1/B_o}{T_o}\right) \tag{23}$$

Fig. 11 is the normalized derivative of Torque with respect to R_m, the mechanical resistance, plotted against R_m.

The resulting normalized derivative is –0.5 at the nominal, full power operating point of R_m=4.0135 which establishes that the mechanical system is conjugately matched (review the Derivative Analysis in Section VIII). Note also that the normalized derivative of –0.5 is unique to the conjugate match condition.

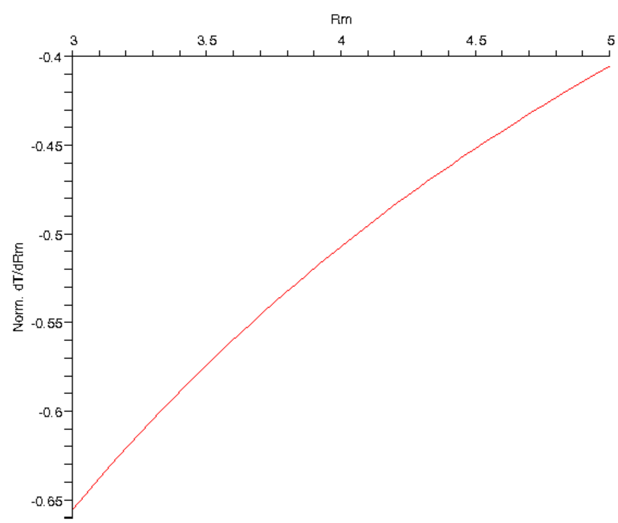

Figure 11. *Normalized derivative of torque with respect to R_m versus R_m.*

I. Summary of the Analysis of an All-Mechanical, Conjugately Matched System

In summary, we have analyzed the mechanical model and have concluded that the power-limited system, consisting of a viscous damper load and a prime-mover/gearbox, is conjugately matched when operating at full power and only when operating at full power. Note also that in this all-mechanical model the efficiency of the overall system is 92%. The viscous damper is merely a "dummy load" representing the useful work that would be done in a more practical load, such as a well pump, a hoist, a ship's propulsion system, etc. We should now be able to accept that the "apparent source impedance" concept applies to the all-mechanical system equally as well as it did to the electro-mechanical system.

1) What About Losses and System Efficiency?

There is no known loss mechanism in the turbine-gearbox portion of the system that would account for the "apparent source impedance" which is easily measured by perturbing the load parameter, R_m, and measuring the system response. The only losses are the 8% efficiency reduction introduced artificially in the gearbox. Clearly, the "apparent source impedance" does not dissipate energy. It is only a manifestation of the action/reaction or automatic adjustment of the system operating point to accommodate a different load – just as was the case in the analysis of the turbine generator system.

2) Is the Prime-Mover/Gearbox the Same in the Two Analyses?

Can we say that the prime-mover/gearbox portion of the system is the same in our two analyses? Yes, they were purposely kept exactly the same. Can we say that the prime-mover/gearbox portion of the system presents the same apparent source impedance in the two analyses? Yes – nothing has changed in that part of the system.

3) Was It Necessary that the Load Torque Characteristic (B) Match the Prime-Mover/Gearbox Torque Characteristic (k_T)?

In this model it was noted that the torque-speed characteristic of the prime-mover/gearbox subsystem has a torque coefficient, k_T, which is equal to the load torque characteristic, B, at the conjugate match point. That is, the ratio of a percent change in torque to a corresponding percent change in speed is the same in both the prime mover and in the load device. Is this is a coincidence brought about by virtue of the fact that both components of the system were given linear torque-speed curves? Since the horsepower curves must intersect at the full power operating point, and since the torque and speed must be equal at the full power operating point, are the torque coefficients also required to be the same? It is not unreasonable for them to have the same torque coefficients, but is it a pre-requisite to obtaining a conjugate match at full power output?

4) Is the Source or Output Impedance of the Prime-Mover/Gearbox a Constant(k_T)?

In the case at hand, the all-mechanical system consisting of a prime-mover/gearbox, has an output impedance that is the inverse of the ratio of change in torque to change in speed, which is the inverse of the dynamic torque coefficient, k_T. That is, the mechanical impedance, R_m, is the impedance that must be matched by the load for a conjugate match to exist and for maximum power output. This was established by the derivative analysis in Section X, subparagraph H.

In Section X, subparagraph G, we showed that the torque versus load impedance curve for the complete system has the same shape as a Norton equivalent circuit in which the output conductance is constant and the load conductance varies. Can we say that the apparent output impedance or source impedance of the prime-mover/gearbox portion of the system is a constant for a given power-limited system? If yes, then that strongly suggests that power-limited systems, whether they are all-mechanical, all electrical, or electro-mechanical, have an apparent source impedance or output impedance that is constant and measurable.

Our conclusion is based first on the shape of the curve and the analysis in Section X, subparagraph G, which showed that the Torque as a function of R_m, (22a), is exactly the same as that for the electrical analog Norton equivalent circuit. Our conclusion is also based on the derivative analysis in Section X, subparagraph H, which demonstrated that a conjugate match occurs at one and only one condition – the condition where the normalized derivative of torque with respect to load resistance is – 0.50 at the nominal, full power output operating point. The conclusion is that the output impedance of a power-limited system is constant, finite and measurable, and it is further concluded that this is a fundamental and unique attribute of a power-limited system.

In summary, we have demonstrated that the concept of a non-dissipative source impedance is also relevant for an all-mechanical system and is therefore not unique to electrical or electronic systems. We have also postulated that the output impedance of power-limited systems (mechanical, electrical, or electro-mechanical) is a constant that must have a conjugate match with the load impedance in order to realize a maximum transfer of power.

XI. Appendix A - Prime Mover Characteristics

Three major categories of prime movers will be discussed – Steam Turbines, Internal Combustion Engines and Hydraulic Turbines.

A. Steam Turbines

Steam Turbines are classified either as impulse class or as reaction class. In either class of turbine superheated steam under high pressure is introduced into the turbine cylinder so as to allow the steam to expand and impart its energy velocity to the turbine blades. See Fig. 12, which is a multi-cylinder or compound turbine [3] being used as the prime-mover for two electrical generators.

In this particular steam turbine example, 1200 p.s.i. steam is first introduced to the upper, high pressure element, which delivers approximately 25% of the total generating capacity of the 50,000 kW system. The high pressure element operates at 3600 RPM. Steam from the high pressure element is then led to the low pressure turbine element, which operates at 1800 RPM and drives its own generator.

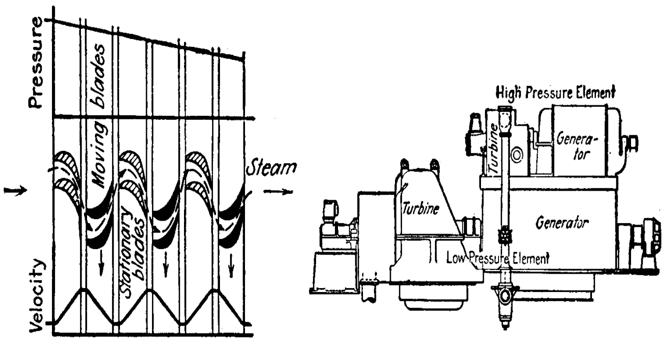

Figure 12. Multi-cylinder or compound turbine

For a given steam generating capability, the steam turbine prime-mover is power-limited. The limiting factor is primarily steam pressure. If the steam turbine and generator system are overloaded, the steam pressure at the input to the High Pressure turbine element decreases and reduces the shaft horsepower being developed by the system.

The source of steam is typically a firebox with its own limitations, such as the rate at which steam can be generated, based on a fuel consumption rate, the temperature of feed water, the back pressure from the load, etc. Even if the steam generating unit had the ability to generate considerably higher steam rates, given time to increase the fuel burning rate, the time it takes to increase its fuel rate and actually deliver higher rates of steam is typically much too long for it to be able to respond to the fluctuations of the turbo-electric unit.

So, for all practical purposes, the steam temperature, pressure and rate of flow are all limited. As a result, a typical steam turbine and electrical generator system operate in a power-limited mode with a horsepower versus speed curve that has a single maximum, such as the example shown in Fig. 13 (The red curve is the power in HP and the green curve is the torque in foot-pounds).

Note that Fig. 13 is not the characteristic curve of the above pictured steam turbine generator set. Instead, it is a set of curves for a hypothetical steam turbine prime-mover.

B. Internal Combustion Engines

Internal combustion engines, such as gasoline and diesel engines, are found in a variety of applications from powering model airplanes all the way up to several thousand horsepower diesels used in diesel-generator sets. The internal combustion engine is inevitably power-limited for several reasons. First is the fuel delivery system. Depending on the type of fuel delivery system, both fuel delivery and air breathing capabilities are either individually limited or limited as a fuel-air mixture. When the throttling device is wide open, the fuel delivery system will max out such as to create a single maximum in the horsepower versus speed curve. See Fig. 14, [4], for the torque and horsepower curves versus speed for a typical automobile engine.

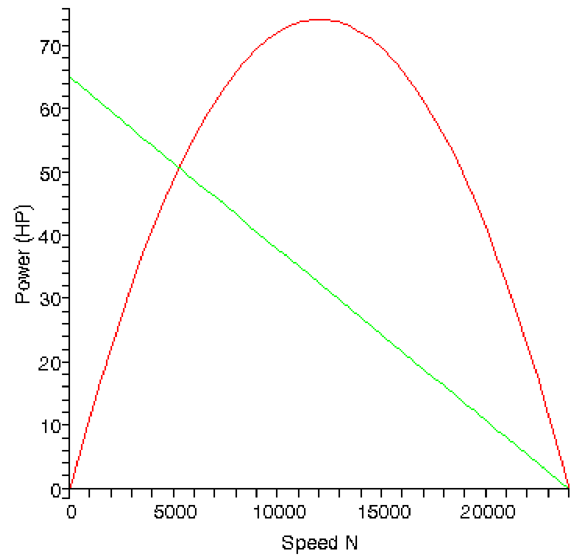

Figure 13. Torque and Horsepower versus speed for a typical steam turbine.

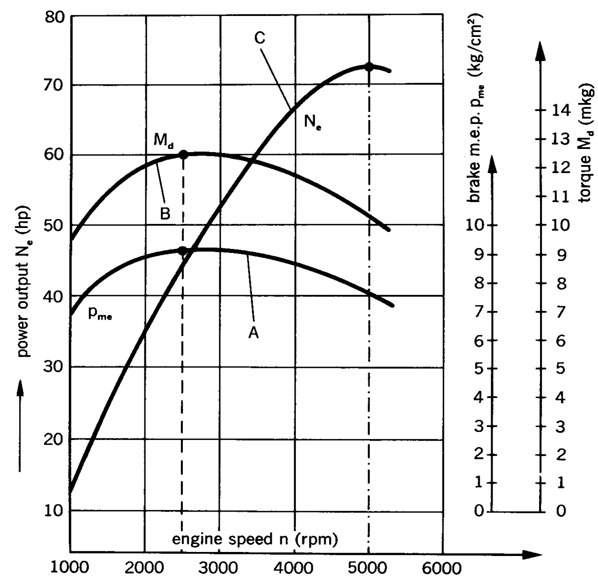

Figure 14. Torque and horsepower versus speed for automotive engine.

The torque curve is curve B and the horsepower curve is curve C. The horsepower curve has only one maximum, and in this example it is just over 70 horsepower and about 73 foot-pounds of torque at 5000 RPM. The torque curve has a shallow slope of about 0.01444 ft-pounds per RPM in the vicinity of the maximum power output point. As the load increases (as in going uphill), engine speed decreases and the operating point seeks a new equilibrium at a lower speed, higher torque and lower horsepower. Conversely, as the load on the engine decreases (as in going downhill), the engine speed increases while the torque and horsepower decrease. This power-limited behavior is common to all internal combustion engines.

C. Hydraulic Turbines

The hydraulic turbine has many characteristics in common with other prime-movers. Fig. 15 shows curves of constant efficiency plotted against speed in RPM (solid curves). The dashed curves are curves of constant gate opening (throttle) plotted against speed and discharge in cubic feet per second. The curve of highest efficiency is of interest to us, because it shows us the nominal speed of the nominal full power output of the device normalized to a 1 foot head as 106 RPM. A real hydraulic turbine will normally operate at a head of dozens of feet, but it is convenient for designers to work with the data normalized to a 1 foot head.

Fig. 16 shows the torque versus speed curves of the same hydraulic turbine as in Fig. 15. If we use the 106 RPM nominal full power speed with the torque curve for the fully open gate, we see that the full power torque is approximately 2.07 pounds with a lever arm of 63 inches or 5.25 feet. Using that point as the full power operating point and making a graphical measurement of the slope of the torque curve at that point, we can approximate the torque for this device and from that the horsepower as a function of speed. The slope of the torque curve is found graphically to be 0.105 foot pound per RPM. If we linearize the torque curve in the vicinity of the full power operating point we see an intercept with the ordinate at 4.22 pounds or 22.15 foot-pounds stall torque.

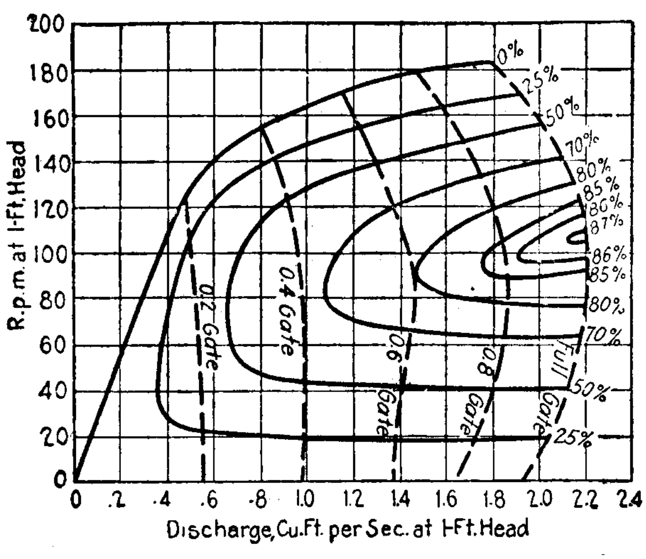

Figure 15. Characteristic curves of hydraulic turbine.

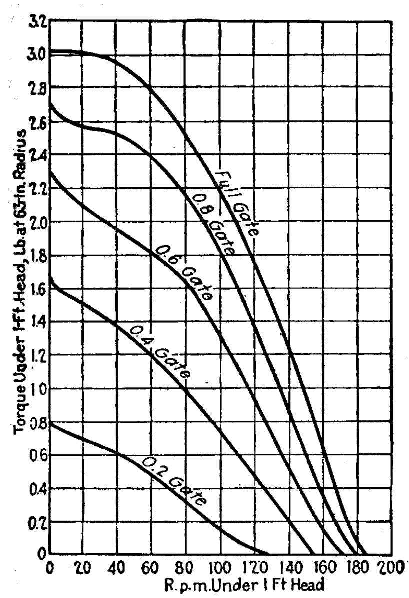

Figure 16. Torque versus speed for hydraulic turbine.

The torque is as follows:

$$T = T_S - k_T * N$$

The power curve is found from the relationship

$$P = k_N * T * N = k_N * (T_S * N - k_T * N * N)$$

where k_N is the conversion factor for converting foot pounds per RPM to Horsepower. Fig. 17 is a plot of the hydraulic turbine horsepower (HP) as a function of the speed N in RPM and based on a stall torque, T_S , of 22.15 foot pounds and a torque coefficient, k_N , of 0.105 foot pounds per RPM. Although we are showing the horsepower curve over the entire speed range, it is only valid for the region within 15-20% of the nominal full power operating speed of 106 RPM. That is the range in which we are interested. Within that range of operation it is clear that the hydraulic turbine is power limited, just as with all other practical prime-movers.

In normal operation, if the turbine is overloaded, its speed immediately decreases to find a new equilibrium point at a lower speed and lower power output and higher torque. Actually, that process would continue right to the stall point. Conversely, if the load on the turbine is decreased the speed immediately increases to find a new equilibrium point at a lower torque and a lower power level. This behavior is consistent with the behavior of the other prime-movers that we have described.

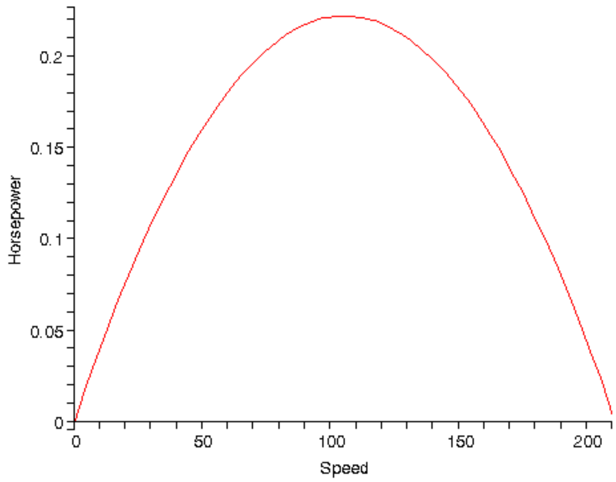

Figure 17. Typical hydraulic turbine horsepower versus speed.

Note that it has a single maximum of output power, which is at approximately 106 RPM.

XII. APPENDIX B –THE EQUATIONS OF LOAD CURRENT AS A FUNCTION OF LOAD RESISTANCE

This appendix will derive the equation for the load current as a function of the load resistance in a Thévenin equivalent circuit.

Consider a series circuit consisting of an emf, E, a source resistance R_S and load resistance R_L. The current flowing in the circuit will be as shown in (24).

$$I = E / (R_S + R_L) \quad (24)$$

We define the Thévenin short circuit current $I_{SC} = E / R_S$ and the source conductance $B_S = 1 / R_S$. Re-arranging (24) to use the short circuit current and the source conductance, we obtain (25).

$$I = (I_{SC} / R_L) / (B_S + (1 / R_L)) \quad (25)$$

Fig. 18 is a plot of the load current, I, against the load resistance R_L for a short circuit current of 50 amps and a source conductance of 0.25 mhos.

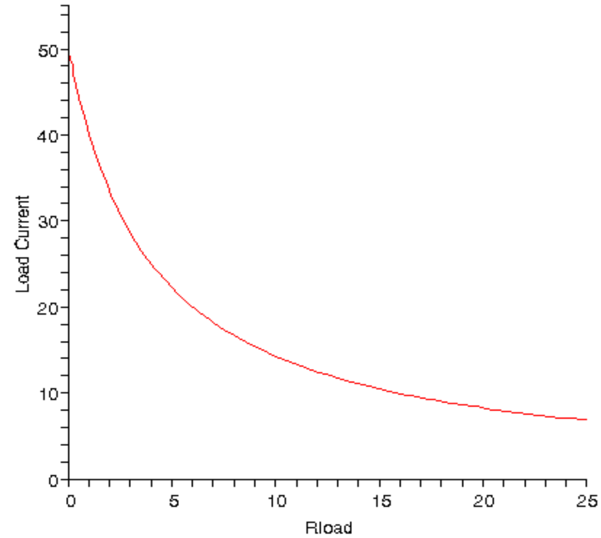

Figure 18. Load current versus load resistance

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